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Math 132: Differential Topology

§ Gauss-Bonnet theorem

Let M be a compact hypersurface (i.e. m-dim'l submanifold) in $\mathbb{R}^{m+1}$.

The <u>Gauss map</u> $g: M \to S^m$ sends each point x in M
$$x \mapsto \vec{n}(x)$$
to the outward pointing unit normal vector $\vec{n}(x) \in S^m$.

Define the <u>volume form</u> vol_M on M to be the m-form on M which evaluates to $\frac{1}{m!}$ on each positively oriented orthonormal basis of T_xM.

Define vol_{S^m} on S^m in the same way.

Since they're top-dim'l forms, at each point $x \in M$, $g^* \mathrm{vol}_{S^m}$ must be a scalar multiple of vol_M; we call that scalar the <u>Gaussian curvature</u> of M at x.

That is, the Gaussian curvature is the function
$$K: M \to \mathbb{R}$$
such that $g^* \mathrm{vol}_{S^m} = K \,\mathrm{vol}_M$.

It measures how much g distorts the volume (and orientation).

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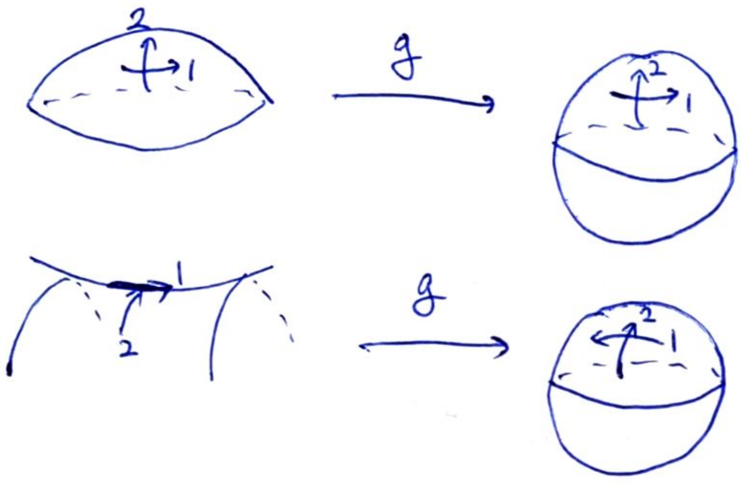
Ex (hypersurfaces in $\mathbb{R}^3$)



$\rightsquigarrow \quad K > 0$

$\rightsquigarrow \quad K < 0$

The Gauss-Bonnet theorem relates the Gaussian curvature (geometric, local) with the Euler characteristic (global, topological):

Thm (Gauss-Bonnet) If M is a compact, even-dim'l hypersurface in $\mathbb{R}^{m+1}$, then

$$\int_M K = \frac{1}{2} \gamma_m \chi(M),$$

where $\gamma_m := \int_{S^m} \mathrm{vol}_{S^m}$ is the volume of the unit m-sphere $S^m \subset \mathbb{R}^{m+1}$

$$\left(\text{i.e. } \gamma_m = \frac{2 \cdot (2\pi)^{\frac{m}{2}}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (m-1)} \right).$$

Rmk The formula is false when M is odd-dim'l, since $\chi(M) = 0$ in that case, while $\int_M K$ can be non-zero.

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proof)

We have $\displaystyle\int_M \kappa \, vol_M \overset{\text{defn of } \kappa}{=} \int_M g^* vol_{S^m} \underset{\text{degree formula}}{=} deg(g) \int_{S^m} vol_{S^m} = deg(g)\, \gamma_m$,

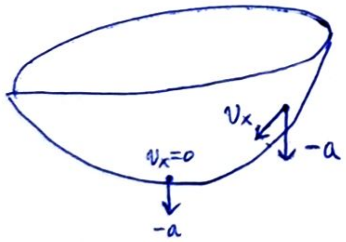
so Gauss-Bonnet thm is equivalent to the claim that

$$\chi(M) = 2\, deg(g).$$

This can be shown using Poincaré-Hopf index thm:

Choose $a \in S^m$ such that $\pm a$ are regular values of g.

Let v be the vectorfield on M whose value v_x at $x \in M$ is the projection of $-a$ onto $T_x M$.



Note, $z \in M$ is a zero of v if and only if $g(z) = \pm a$.

It follows that v only has finitely many zeros, and

$$\chi(M) = \sum_{v(z)=0} ind_z v = \sum_{\substack{v_z=0,\\ g(z)=a}} ind_z v + \sum_{\substack{v_x=0,\\ g(z)=-a}} ind_z v$$

$$\underset{dv_z = \pm dg_z}{=} deg(g) + (-1)^m deg(g) = (1+(-1)^m)\, deg(g)$$

$$= 2\, deg(g). \quad \blacksquare$$